

# BOREL DEGENERATIONS OF ARITHMETICALLY COHEN-MACAULAY CURVES IN $\mathbb{P}^3$

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**ABSTRACT.** We investigate Borel ideals on the Hilbert scheme components of arithmetically Cohen-Macaulay (ACM) codimension two schemes in  $\mathbb{P}^n$ . We give a basic necessary criterion for a Borel ideal to be on such a component. Then considering ACM curves in  $\mathbb{P}^3$  on a quadric we compute in several examples all the Borel ideals on their Hilbert scheme component. Based on this we conjecture which Borel ideals are on such a component, and for a range of Borel ideals we prove that they are on the component.

## INTRODUCTION

The ideal of any subscheme in a projective space  $\mathbb{P}^n$  may be degenerated through coordinate changes, to a Borel fixed monomial ideal (henceforth called a Borel ideal). So any component of the Hilbert scheme of subschemes of  $\mathbb{P}^n$  contains a Borel ideal.

Borel ideals in characteristic zero have nice combinatorial descriptions. Borel ideals are also the most degenerate of all ideals in the sense that if we degenerate a Borel ideal  $J$  to another monomial ideal  $J'$  through coordinate changes, then  $J'$  is simply obtained from  $J$  through a permutation of the variables. Put in another way, the  $GL(n+1)$ -orbit on the Hilbert scheme of a Borel ideal, is closed. This raises the problem of investigating and finding these the most degenerate ideals on a component. For instance A.Reeves, [12], asks if the set of Borel ideals on a component characterizes the component.

The interest in the geography of Borel ideals on the Hilbert scheme may be said to date back to Hartshorne's proof of the connectedness of the Hilbert scheme. Proceeding through a succession of distractions and degenerations one may proceed from any Borel ideal to the lex segment Borel ideal. Surprisingly it was shown that this ideal is a smooth point on the Hilbert scheme, [11], thus identifying a distinguished component, the lex segment component of the Hilbert scheme. P.Lella in [10] shows how Borel ideals may be connected by irreducible rational curves on the Hilbert scheme and so provides insight into the network

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of Borel ideals on the Hilbert scheme, and in particular provides a new proof of its connectedness.

In this paper we consider the Hilbert scheme components of arithmetically Cohen-Macaulay (ACM) subschemes of codimension two in  $\mathbb{P}^n$ . These are characterized by their homogeneous ideal in  $k[x_0, \dots, x_n]$  having the shortest possible resolution by free  $S$ -modules, of length two. The Hilbert scheme components of ACM codimension two subschemes are well classified. In particular there is a one-to-one correspondence between such components and ACM Borel fixed ideals of codimension two for the ordering  $x_0 > x_1 > \dots$ . They have the following form

$$J(a; \mathbf{b}) = (x_0^a, x_0^{a-1}x_1^{b_1}, x_0^{a-2}x_1^{b_2}, \dots, x_1^{b_a})$$

where  $0 = b_0 < b_1 < \dots < b_a$ .

We find two basic necessary conditions for a Borel ideal to be on the component of the above ideal. The first condition is:

**Theorem 1.5.** *Let  $J$  be a Borel ideal on the Hilbert scheme component of  $J(a; \mathbf{b})$ . Let  $d_s = \sum_{i=1}^s b_i$ . Then  $x_0^{a-s}x_1^{d_s}$  is in  $J$  for each  $s = 0, \dots, a$ .*

The second condition is standard and follows by the semi-continuity of the cohomology of coherent sheaves.

**Condition 2.** If  $J$  is a saturated Borel ideal on the component of  $J(a; \mathbf{b})$ , then the Hilbert functions  $h_J(d) \geq h_{J(a; \mathbf{b})}(d)$  for all  $d$ .

We then proceed to investigate closer what are the Borel ideals on specific components. This is a hard task and to obtain reasonably comprehensive results we restrict ourselves to the case of ACM curves on quadrics in  $\mathbb{P}^3$ . The components of such curves correspond to Borel ideals

$$J(l, m) = (x^2, xy^l, y^{l+m})$$

where  $l, m \geq 1$  and  $S = k[x, y, z, w]$ . In several example cases, for the following values of  $(l, m)$ :

$$(1, 3), (2, 2), (3, 1), (3, 3),$$

we find by computation all Borel ideals on this component. For instance when  $(l, m) = (3, 3)$  there are 989 Borel ideals with the same Hilbert polynomial as  $J(3, 3)$ , but only 45 of these are on the component of  $J(3, 3)$ .

In all the computed cases the only obstructions we have found for a Borel ideal to be on the component of  $J(l, m)$  are given by Theorem 1.5 and Condition 2. We therefore make the following.

**Conjecture 2.2.** *If a saturated Borel ideal ideal has the same Hilbert polynomial as  $J(l, m)$ , then it is on the component of  $J(l, m)$  if and only if it fulfills the criteria of Theorem 1.5 and Condition 2.*

We then exhibit many classes of Borel ideals that are on the component of  $J(l, m)$ . To do this we consider explicit families of curves defined by the  $2 \times 2$ -minors of the matrix

$$\begin{bmatrix} x & y^l & -F \\ 0 & x & y^m \end{bmatrix}.$$

By specializing  $F$  in various ways we get various Borel ideals as specializations. In Sections 3, 4, and 5, we investigate in particular the cases when  $l = 1, 2$ , and  $3$  and show many classes of Borel ideals to be on the component of  $J(l, m)$ . In the last Section 6 we give a class of Borel ideals for general  $l$  which is on the component.

The organization of the paper is as follows. In Section 1 we recall basic facts about Hilbert scheme components containing ACM codimension two subschemes of  $\mathbb{P}^n$  and prove the basic condition Theorem 1.5 on Borel ideals on such a component. In Section 2 we study example cases and compute all the Borel ideals on the component of  $J(l, m)$  in the range of  $(l, m)$  stated above. We also conjecture what are the Borel ideals on the component of  $J(l, m)$ . In Section 3 we state our main theorems of sufficient conditions for a Borel ideal to be on the component of  $J(l, m)$ . In Section 4 we give the families of ideals that we degenerate, and in Section 5 we prove the results. In Section 6 we exhibit a general class of Borel ideals on the component of  $J(l, m)$ .

## 1. A NECESSARY CONDITION ON BOREL DEGENERATIONS

In this section we consider components of the Hilbert scheme whose general point corresponds to arithmetically Cohen-Macaulay (ACM) schemes of codimension two. These components are well classified. We give a necessary condition for a Borel ideal to be on such a component.

### 1.1. ACM codimension two components of the Hilbert scheme.

A subscheme  $X \subseteq \mathbb{P}_k^n$  where  $k$  is a field, is *arithmetically Cohen-Macaulay (ACM)* if its homogeneous ideal  $I = I_X$  in the polynomial ring  $S = k[x_0, \dots, x_n]$  has a minimal free resolution of length two

$$I \leftarrow G_0 \xleftarrow{\phi} G_1$$

where  $G_0$  and  $G_1$  are graded free  $S$ -modules.

Let  $H$  be the Hilbert scheme corresponding to the Hilbert polynomial of the quotient ring of  $I$ . There is a universal family of schemes

$$\begin{array}{ccc} Z & \subseteq & H \times \mathbb{P}^n \\ & & \downarrow \\ & & H \end{array}$$

flat over  $H$ , and let  $\mathcal{I}_Z$  be its ideal sheaf in  $\mathcal{O}_{H \times \mathbb{P}^n}$ . If we have a morphism from an affine ring  $\text{Spec } B \rightarrow H$  we may pull back  $\mathcal{I}_Z$  and get an ideal sheaf  $\mathcal{I}_B$  in  $\mathbb{P}_B^n$ . Denote by  $I_B$  the graded ideal in  $B[x_0, \dots, x_n]$  of global sections  $\bigoplus_{d \in \mathbb{Z}} \Gamma(\mathbb{P}_B^n, \mathcal{I}_B(d))$ . Note that since  $\text{Spec } B$  is affine,

the sheafification of  $(I_B)_d$  over  $\text{Spec } B$  is the pushdown  $p_*\mathcal{I}_B(d)$  by the natural map  $p : \mathbb{P}_B^n \rightarrow \text{Spec } B$ . The graded ideal  $I_B$  will in general not be a flat family of ideals over  $\text{Spec } B$ . We shall however see that in our situation, there is an open subset  $\text{Spec } B$  of the Hilbert scheme  $H$  such that  $I_B$  becomes a flat family of ACM codimension two ideals with the same resolution as  $I$ .

**Proposition 1.1.** *Let  $n \geq 3$  and*

$$I \leftarrow G_0 \leftarrow G_1$$

*be the minimal free resolution of an ACM ideal  $I$  of codimension two, corresponding to a point  $\mathfrak{i}$  on the Hilbert scheme  $H$ . Then there is an open affine subset  $U = \text{Spec } B \subseteq H$  of  $\mathfrak{i}$  such that the ideal of graded global sections  $I_B$  defined above is flat over  $B$  and has a resolution*

$$I_B \leftarrow G_0 \otimes_k B \xleftarrow{\phi_B} G_1 \otimes_k B$$

*whose fibre at the point  $\mathfrak{i}$  is the resolution of  $I$ .*

*Proof.* Let  $\text{Spec } B \subseteq H$  be an open affine neighbourhood of  $\mathfrak{i}$ . We get an ideal sheaf  $\mathcal{I}_B$  in  $\mathbb{P}_B^n$ . For each  $i = 1, \dots, n$  there is a  $d(i)$  such that  $R^i p_* \mathcal{I}_B(d)$  vanishes for  $d \geq d(i)$ , by [7, Thm. III.5.2]. Let  $d_0$  be the maximum of the  $d(i)$ .

By the Cohomology and Base Change Theorem (CBCT) [7, Thm. III.12.11] part b. we get that  $p_* \mathcal{I}_B(d)$  is a locally free (or flat)  $B$ -module for  $d \geq d_0$ , and that all the cohomology modules  $H^i(\mathbb{P}^n, \mathcal{I}_{k(\mathfrak{b})}(d))$  vanish for  $\mathfrak{b} \in \text{Spec } B$  when  $i > 0$  and  $d \geq d_0$ . Also note that  $p_* \mathcal{I}_B(d)$  and all the cohomology modules  $H^0(\mathbb{P}_{k(\mathfrak{b})}^n, \mathcal{I}_{k(\mathfrak{b})}(d))$  vanish for  $d < 0$  since they are submodules of  $p_* \mathcal{O}_{\mathbb{P}_B^n}(d)$  and  $k(\mathfrak{b})[x_0, \dots, x_n]_d$  respectively.

The fibre ideal  $\mathcal{I}_{k(\mathfrak{i})}$  has  $I$  as its associated graded ideal. Since  $n \geq 3$ , by running the long exact cohomology sequence on the sheafification of the resolution of  $I$ , we get the vanishing of  $H^1(\mathbb{P}_{k(\mathfrak{i})}^n, \mathcal{I}_{k(\mathfrak{i})}(d))$  for all  $d$ . By semi-continuity of cohomology, there is an open subset  $\text{Spec } B_f$  of  $\text{Spec } B$ , containing  $\mathfrak{i}$ , such that the  $H^1(\mathbb{P}_{k(\mathfrak{b})}^n, \mathcal{I}_{k(\mathfrak{b})}(d))$  vanish for  $\mathfrak{b}$  in  $\text{Spec } B_f$  and  $d = 0, \dots, d_0$ . But then we know by the above that they also vanish for all  $d \geq 0$ .

By part a. of CBCT the maps

$$R^1 p_*(\mathcal{I}_{B_f}(d)) \otimes_B k(\mathfrak{b}) \rightarrow H^1(\mathbb{P}_{k(\mathfrak{b})}^n, \mathcal{I}_{k(\mathfrak{b})}(d)),$$

being surjective, will be isomorphisms for  $\mathfrak{b}$  in  $\text{Spec } B_f$ . Hence by Nakayama's lemma, we obtain the vanishing of  $R^1 p_*(\mathcal{I}_{B_f}(d))$  for all  $d \geq 0$ .

By part b. of CBCT (applied when  $i = 1$ ) the maps

$$R^0 p_*(\mathcal{I}_{B_f}(d)) \otimes_{B_f} k(\mathfrak{b}) \rightarrow H^0(\mathbb{P}_{k(\mathfrak{b})}^n, \mathcal{I}_{k(\mathfrak{b})}(d))$$

are also surjective for  $d \geq 0$  and  $\mathfrak{b}$  in  $\text{Spec } B_f$ . Applying CBCT again (when  $i = 0$ ), these maps are isomorphisms, and  $R^0 p_*(\mathcal{I}_{B_f}(d))$  is locally

free in a neighbourhood of any such  $\mathfrak{b}$ , and so on all of  $\text{Spec } B_f$ . Hence the graded ideal  $I_{B_f}$  is a flat  $B_f$ -module.

For ease of notation denote  $B_f$  further on simply as  $B$ . We may lift the start of the resolution of  $I$  to a diagram

$$\begin{array}{ccc} I_B & \xleftarrow{p_B} & G_0 \otimes_k B \\ \downarrow & & \downarrow \\ I & \xleftarrow{p} & G_0. \end{array}$$

The cokernel of  $p_B$  vanishes in the fibre at  $\mathfrak{i} \in \text{Spec } B$ . Since we are in a noetherian setting,  $\text{coker } p_B$  has a finite set of generators over  $B[x_0, \dots, x_n]$ . By taking a suitable localization  $B_f$  (by abuse of notation we still denote it by  $B$ ), all these generators vanish and so  $p_B$  is surjective. Since  $I_B$  and  $G_0 \otimes_k B$  are  $B$ -flat, the kernel of  $p_B$  will be  $B$ -flat, hence

$$0 \leftarrow I_B \otimes_B k(\mathfrak{i}) \leftarrow (G_0 \otimes_k B) \otimes_B k(\mathfrak{i}) \leftarrow (\ker p_B \otimes_B k(\mathfrak{i})) \leftarrow 0$$

is exact and so the right term in this sequence is equal to  $\ker p$ . We may now continue the process and lift to a diagram

$$\begin{array}{ccccc} I_B & \xleftarrow{p_B} & G_0 \otimes_k B & \xleftarrow{\phi_B} & G_1 \otimes_k B \\ \downarrow & & \downarrow & & \downarrow \\ I & \xleftarrow{p} & G_0 & \xleftarrow{\phi} & G_1 \end{array}$$

and by using flatness we see that the upper sequence is exact after localizing  $B$  suitably.  $\square$

**Corollary 1.2.** *If a component of the Hilbert scheme contains an ACM ideal of codimension two, the general point on the component will be an ACM codimension two ideal with the same Hilbert function.*

Such a component will be called an ACM codimension two component of the Hilbert scheme.

**1.2. Borel ideals.** We assume in the following that our field  $k$  has characteristic zero. A monomial ideal is called a Borel fixed ideal, or simply Borel ideal, if whenever a monomial  $x_j m \in J$  and  $i < j$ , then the monomial  $x_i m \in J$ . This is equivalent to  $J$  being invariant for the Borel subgroup of  $GL(n+1)$  consisting of the upper triangular matrices (when the linear forms have a basis  $x_0, x_1, \dots, x_n$ ). The Borel ideals which are ACM of codimension two are easy to describe. They are given by their minimal generators as

$$(1) \quad J(a, \mathbf{b}) = (x_0^a, x_0^{a-1} x_1^{b_1}, x_0^{a-2} x_1^{b_2}, \dots, x_1^{b_a})$$

where  $0 = b_0 < b_1 < \dots < b_a$ .

If  $I$  is any homogeneous ideal, and we take its generic initial ideal  $\text{gin}(I)$  for the revlex order, then  $\text{gin}(I)$  will be a Borel ideal [5, 15.9],

and ii) have the same depth as  $I$ , by [2]. Hence if  $I$  is ACM of codimension two, its generic initial ideal will be a Borel ACM codimension two ideal and so have the form (1) above.

Let us now collect the following facts.

1. Each ACM codimension two component contains a Borel ideal of the type  $J(a, \mathbf{b})$  (by the above argument).
2. Such a Borel ideal is a smooth point on the Hilbert scheme, [6], and hence is on a single component.
3. Distinct ideals  $J(a, \mathbf{b})$  in (1) are on distinct components. This follows by the above Corollary 1.2 since it is easy to see that distinct pairs  $(a, \mathbf{b})$  will give distinct Hilbert functions.

In conclusion we get the following well known fact.

**Proposition 1.3.** *There is a one-to-one correspondence between ACM codimension two components of the Hilbert scheme and ideals  $J(a, \mathbf{b})$ .*

Now we shall investigate Borel fixed ideals on the component of  $J(a, \mathbf{b})$ . Let us start with an example.

*Example 1.4.* Twisted cubic curves in  $\mathbb{P}^3$  are ACM curves with Hilbert polynomial  $3d + 1$ . The corresponding Borel ideal is

$$J = (x_0^2, x_0x_1, x_1^2).$$

It is not difficult to show that the Borel ideal

$$I = (x_0^2, x_0x_1, x_0x_2, x_1^3)$$

is on the component. The ideal

$$K = (x_0, x_1^3x_2, x_1^4)$$

is the lex segment ideal with Hilbert polynomial  $3d + 1$ . It is a smooth point on the Hilbert scheme and the single component containing it is different from the one of  $J$ .

Another way to see this is the general theorem below which implies that if a Borel ideal is a degeneration of the ideal of a twisted cubic curve then it must contain  $x_1^3$ .

**Theorem 1.5.** *Let  $J$  be a Borel ideal on the Hilbert scheme component of  $J(a, \mathbf{b})$ . Let  $d_s = \sum_{i=1}^s b_i$ . Then  $x_0^{a-s}x_1^{d_s}$  is in  $J$  for each  $s = 0, \dots, a$ .*

*Proof.* We apply Proposition 1.1 to the ideal  $I = J(a, \mathbf{b})$ . By the Hilbert-Burch theorem the ideal  $I_B$  of Proposition 1.1 is generated by the minors of the matrix  $\phi_B$ . Denote these minors as

$$F_0, F_1, \dots, F_a.$$

Let  $A$  be the local ring at the point in  $\text{Spec } B$  corresponding to  $I$ . Considering the  $F_i$ 's over this local ring we may write

$$F_i = x_0^{a-i}x_1^{b_i} + \sum_j c_{i,j}M_{i,j}$$

where the  $M_{i,j}$  are monomials in  $k[x_0, \dots, x_n]$  of degree  $a - i + b_i$  and the  $c_{i,j}$  are in the maximal ideal of  $A$ . By subtracting multiples of  $F_0$  we may assume that  $x_0^a$  is the highest power of  $x_0$  occurring in any of the  $F_i$ . Then there will be an open subset  $\text{Spec } B_f \subseteq \text{Spec } B$  such that considering the  $F_i$  over  $B_f$  we may for each  $i$  write

$$F_i = x_0^{a-i} x_1^{b_i} + \sum_{j=0}^a x_0^{a-j} E_{i,j}$$

where the  $E_{i,j}$  are polynomials in the variables  $x_1, \dots, x_n$ . We can then write the transpose

$$[F_0, \dots, F_a]^t = W \cdot [x_0^a, \dots, x_0, 1]^t$$

where  $W$  is an  $(a+1) \times (a+1)$  matrix with entries  $E_{i,j}$  in position  $(i, j)$  when  $i \neq j$  and  $x_1^{b_i} + E_{i,i}$  when  $i = j$ . For simplicity of notation we denote  $B_f$  further on by  $B$ .

Let  $W_s$  be the upper left  $(s+1) \times (s+1)$  submatrix of  $W$ . Modulo the  $B[x_1, \dots, x_n]$ -submodule  $\oplus_{i=0}^{a-s-1} x_0^i B[x_1, \dots, x_n]$  we get

$$[F_0, \dots, F_s]^t \equiv W_s \cdot [x_0^a, \dots, x_0^{a-s}]^t.$$

By considering the diagonal of  $W_s$  we see that its determinant has degree  $d_s = \sum_{i=1}^s b_i$ . Let  $V_s$  be the matrix of cofactors of  $W_s$ , so that  $V_s \cdot W_s = (\det W_s) \cdot I$ . Then

$$V_s \cdot [F_0, \dots, F_s]^t \equiv (\det W_s) \cdot [x_0^a, \dots, x_0^{a-s}]^t.$$

Hence the last entries in these products are

$$G_s := \sum_{i=0}^s (V_s)_{s,i} F_i \equiv x_0^{a-s} \cdot (\det W_s).$$

Note that the right side has degree  $a - s + d_s$ .

Now any saturated monomial ideal  $J$  corresponding to a point in the closure of the open subset  $U = \text{Spec } B$  must contain some monomial in  $G_s$ . This will be a monomial of degree  $a - s + d_s$  and with  $x_0$ -degree  $\leq a - s$ . So if  $J$  is Borel fixed for the ordering  $x_0 > x_1 > \dots > x_n$  of the variables, it must then contain  $x_0^{a-s} x_1^{d_s}$ .  $\square$

## 2. EXAMPLES AND CONJECTURES

We now consider ACM curves in  $\mathbb{P}^3$  and their Hilbert scheme components. This section will systematically investigate example cases where the curves on such a component is on a quadric and find all Borel ideals on such a component.

**2.1. ACM ideals on a quadric.** Since now  $n = 3$  we write the polynomial ring as  $S = k[x, y, z, w]$ . Ordering the variables as  $x > y > z > w$ , an ACM Borel ideal on a quadric may then be written as

$$J(l, m) = (x^2, xy^l, y^{l+m}).$$

Denote its Hilbert scheme component as  $H(l, m)$ . The resolution of  $J(l, m)$  is

$$J(l, m) \leftarrow \begin{array}{c} [x^2, -xy^l, y^{l+m}] \\ S(-2) \oplus S(-l-1) \oplus S(-l-m) \end{array} \leftarrow \begin{array}{c} \begin{bmatrix} y^l & 0 \\ x & y^m \\ 0 & x \end{bmatrix} \\ S(-l-2) \oplus S(-l-m-1). \end{array}$$

In our examples we shall repeatedly consider ACM ideals on the component  $H(l, m)$  which are generated by the  $2 \times 2$  minors of the matrix

$$(2) \quad A(F) = \begin{bmatrix} x & y^m & -F \\ 0 & x & y^l \end{bmatrix}$$

where  $F$  is a polynomial in  $k[x, y, z, w]$ , homogeneous of degree  $l+m-1$  in the  $x, y, z, w$ . Performing row and column operations on the matrix, we may assume that

$$(3) \quad F = y^{l-1}F_m + y^{l-2}F_{m+1} + \cdots + y^0F_{m+l-1}.$$

The  $2 \times 2$ -minors of the matrix are

$$(4) \quad x^2, \quad xy^l, \quad G := xF + y^{m+l}.$$

By multiplying  $G$  with  $y^l$  we see that the ideal generated by these minors will also contain  $y^{2l+m}$ .

*Example 2.1.* The case  $l = 1$  and  $m = 3$ . The Hilbert polynomial is  $5t - 2$ . There are 7 (saturated) Borel ideals on the Hilbert scheme  $\text{Hilb}_{5t-2}^3$  and these are:

- $J_1 = (x, y^6, y^5z^3)$
- $J_2 = (x, y^7, y^6z, y^5z^2)$
- $J_3 = (x^2, xy, xz, y^6, y^5z^2)$
- $J_4 = (x^2, xy, xz^2, y^6, y^5z)$
- $J_5 = (x^2, xy, xz^3, y^5)$
- $J_6 = (x^2, xy^2, xyz, xz^2, y^5)$
- $J_7 = (x^2, xy, y^4).$

Which of these Borel ideals are on the component  $H(1, 3)$  of the ACM Borel ideal  $J_7$ ? By Theorem 1.5 a necessary condition is that  $y^5$  is in the Borel ideal. So the first four  $J_1, J_2, J_3$  and  $J_4$  are not on the component  $H(1, 3)$ . On the other hand by semi-continuity of the cohomology of sheaves in flat families [7, III.12], any saturated ideal on the component  $H(1, 3)$  must have a Hilbert function which is greater or equal to that



of  $J_7$ . Since we find  $h_{J_6}(2) = 1$  and  $h_{J_7}(2) = 2$ , we see that  $J_6$  cannot be on the component  $H(1, 3)$ . This leaves only the possibility of  $J_5$  (and of course  $J_7$ ) to be on  $H(1, 3)$ .

By letting  $F = z^3$  in (4) we get the ideal  $I = (x^2, xy, y^4 + xz^3)$  corresponding to a smooth point on  $H(1, 3)$ . The initial ideal of  $I$  with respect to the lexicographic order is  $J_5$  (and with respect to the reverse lexicographic order it is  $J_7$ ). Hence the ideal  $J_5$  is on the component  $H(1, 3)$ . In conclusion we have established that the only Borel ideals on  $H(1, 3)$  are  $J_5$  and  $J_7$ .

**2.2. Conjectures.** In the above example there were two obstructions which ruled out a saturated Borel ideal from being on the ACM component. By Theorem 1.5 we must have:

C1. If a (saturated) Borel ideal  $J$  is on the component  $H(l, m)$  then  $y^{2l+m} \in J$ .

By semi-continuity of the cohomology of coherent sheaves [7, III.12] we must have:

C2. If a (saturated) Borel ideal  $J$  is on the component  $H(l, m)$  then the hilbert function  $h_J(d) \geq h_{J(l,m)}(d)$  for all  $d \geq 0$ .

In all the examples we have computed, these are the only two obstructions we have found. We therefore make the following.

**Conjecture 2.2.** *If a saturated Borel ideal  $J$  has the same Hilbert polynomial as  $J(l, m) = (x^2, xy^l, y^{l+m})$ , then it is on the Hilbert scheme component of  $J(l, m)$  if and only if i)  $y^{2l+m} \in J$  and ii) the hilbert function  $h_J(d) \geq h_{J(l,m)}(d)$  for all  $d \geq 1$ .*

Now we proceed to consider more examples. As soon as we have eliminated all Borel ideals not fulfilling C1. and C2., the challenge is to show that the remaining ideals are on the component  $H(l, m)$ . We shall do this in several examples, illustrating computational arguments and techniques. The first technique is to take a suitably general ideal and take its initial ideals for various term orders. Given integers  $v_1 > v_2 > v_3 > v_4$ , a term order  $\prec_{[v_1, v_2, v_3, v_4]}$  may be specified by the matrix (see [8, Sec. 1.4]):

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ v_1 & v_2 & v_3 & v_4 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

This means that two monomials  $\mathbf{x}^{\mathbf{a}} > \mathbf{x}^{\mathbf{b}}$  if  $M \cdot \mathbf{a} > M \cdot \mathbf{b}$  in the lex order on four-tuples.

*Example 2.3.* The case  $l = 2$  and  $m = 2$ . The Hilbert polynomial is  $6t - 3$ . By using the applet BorelGenerator [9] we may get a list of all Borel ideals with this Hilbert polynomial, and there are 31 such.

Only seven of them fulfill the conditions C1. and C2. These are (as numbered by the applet of loc.cit.):

- $J_{21} = (x^2, xy, y^6, xz^6),$
- $J_{22} = (x^2, xy^2, xyz, y^6, xz^5),$
- $J_{23} = (x^2, xy^2, xyz^2, xz^4, y^6),$
- $J_{27} = (x^2, xy, y^6, y^5z^2),$
- $J_{28} = (x^2, xy^2, xyz, y^6, y^5z),$
- $J_{29} = (x^2, xy^2, xyz^2, y^5),$
- $J_{31} = (x^2, xy^2, y^4).$

A general curve on this ACM component will be a complete intersection of a quadric  $Q$  and a cubic  $C$ . Let  $I = (Q, C)$  where the two forms are chosen generic (i.e. randomly). By Galligos theorem [5, Theorem 15.20] we know that the initial ideal of  $I$  for any term order is Borel fixed. By considering various term orders we find the following initial ideals:

- $\text{in}(I, \prec_{[43,9,2,1]}) = (x^2, xy^2, xyz^2, w^2zyx, y^6, w^4yx, xz^6)$  whose saturation is  $J_{21}$ ;
- $\text{in}(I, \prec_{[17,4,1,0]}) = (x^2, xy^2, xyz^2, w^2zyx, xz^5, y^6)$  whose saturation is  $J_{22}$ ;
- $\text{in}(I, \prec_{[16,4,2,0]}) = J_{23}$ ;
- $\text{in}(I, \prec_{[51,13,2,1]}) = (x^2, xy^2, xyz^2, w^2zyx, w^4yx, y^6, y^5z^2)$  whose saturation is  $J_{27}$ ;
- $\text{in}(I, \prec_{[45,12,2,1]}) = (x^2, xy^2, xyz^2, w^2zyx, y^5z, y^6)$  whose saturation is  $J_{28}$ ;
- $\text{in}(I, \prec_{[38,11,2,1]}) = J_{29}$ .

It is also possible to obtain the Borel ideals above by letting  $I$  be the ideal generated by the  $2 \times 2$ -minors of the matrix  $A(F)$  of (2) where  $F$  is a general form of degree 3 of the form (3). Using exactly the same terms orders as above we find exactly the same initial ideals.

Two notable features of the above example are.

1. Each Borel ideal is the limit of ideals which are generated by the  $2 \times 2$ -minors of the matrix  $A(F)$ . In particular all these ideals contain  $x^2, xy^l$  and  $y^{2l+m}$ .
2. The ideal that we degenerate is obtained by a general choice (either  $Q$  and  $C$ , or  $F$ ).

We shall see that in all our examples we are able to do as in 1. However we are not always able to do as in 2.

**Conjecture 2.4.** *Given a saturated Borel ideal  $J$  on the Hilbert scheme component of the Borel ideal  $J(l, m) = (x^2, xy^l, y^{l+m})$ . Then there is a family of ideals generated by the  $2 \times 2$ -minors of matrices  $A(F)$  of (2), which specialize to a monomial ideal whose saturation is  $J$ .*

**2.3. Segment ideals.** We now illustrate a further technique. If  $J$  is a monomial ideal in  $S = k[x_0, \dots, x_n]$ , its graded piece  $J_t$  is a *segment*

if there is a term order  $\prec$  such that the  $d = \dim_k J_t$  monomials in  $J_t$  are the first  $d$  monomials in  $S_t$  for this term order. It is easy to see, [4, Lem. 3.2] that if  $I_t$  is a segment for  $\prec$ , then  $I_s$  is also a segment for  $\prec$  when  $s < t$ .

Two cases are particularly noteworthy, [4]. One case is when  $t$  is the regularity of  $J$  (see [5, 20.5] for the definition of regularity). For a Borel ideal this is simply the degree of the largest generator of  $J$ . In this case  $J$  is called a reg-segment ideal. The other case is when  $t$  is the Gotzmann number  $r$  of  $J$ . This number depends only on the Hilbert polynomial  $p(t)$  of  $J$  and it is the largest regularity an ideal with Hilbert polynomial  $p(t)$  can have. It is the regularity of the lex segment ideal with this Hilbert polynomial. In this case  $J$  is called a Hilbert segment ideal.

Now let  $I$  be any ideal with the same Hilbert polynomial as  $J$ . We consider the subspace  $I_t \subseteq S_t$ . Choose the monomials in  $S_t$  as a basis for  $S_t$  and project the  $d$ -dimensional space  $I_t$  onto the  $d$ -dimensional space which has the terms in  $J_t$  as a basis. If this map is an isomorphism, then the initial terms of  $I_t$  for the term order  $\prec$  will be the terms in  $J_t$ , since  $J_t$  is a segment. Another way to phrase this is via Plücker coordinates. Denote by  $N$  the dimension of  $S_t$  which is  $\binom{n+t}{t}$ , and let  $M = \binom{N}{d} - 1$ . We then have the Plücker embedding of the Grassmannian of  $d$ -dimensional subspaces of  $S_t$  into projective space  $\mathbb{G}(d, N) \hookrightarrow \mathbb{P}^M$ . The above may be phrased as saying that the Plücker coordinate of  $I$  corresponding to the terms in  $J_t$ , is nonzero.

By applying the below to ACM ideals  $I$  of codimension two, we get a criterion for a Borel ideal  $J$  to be on the ACM component of  $I$ .

**Lemma 2.5.** *Let  $J$  be a saturated Borel ideal with regularity  $m_0$ , and  $I$  a homogeneous ideal with the same Hilbert polynomial as  $J$ . Let  $t \geq m_0$ .*

*a. If  $J_t$  is a segment and the Plücker coordinate of  $I$  with respect to the terms in  $J_t$  are nonzero, the saturation of the initial ideal  $\text{in}(I)$  is  $J$ .*

*More generally we may let  $s, t \geq m_0$ .*

*b. If  $J_t$  is a segment, and the Plücker coordinate of  $I$  with respect to the terms in  $J_s$  are nonzero, the saturation of the initial ideal  $\text{in}(I)$  is  $J$ .*

*In particular  $I$  and  $J$  are on the same component.*

*Proof.* a. Clearly the hypothesis implies that  $\text{in}(I)_t = J_t$ . Since  $J_{\geq m_0}$  is generated by the elements of degree  $m_0$ , clearly  $\text{in}(I)_{\geq t}$  contains  $J_{\geq t}$ . Therefore  $\text{in}(I)^{\text{sat}} \supseteq J^{\text{sat}} = J$  and so there is an exact sequence

$$0 \rightarrow \ker p \rightarrow S/J \xrightarrow{p} S/\text{in}(I)^{\text{sat}} \rightarrow 0.$$

Since the right terms have the same Hilbert polynomials, the kernel  $\ker p$  must be finite dimensional. But since  $S/J$  does not have the

maximal graded ideal in  $S$  as an associated prime ( $J$  being saturated),  $\ker p$  must be zero.

b. The Plücker coordinate of  $I$  with respect to  $J_s$  is nonzero. So the projection of  $I_s$  onto the space of monomials in  $J_s$  is surjective. We show (at the end of the proof) that this implies that the projection of  $I_e$  onto the space of monomials in  $J_e$  is surjective for all  $e \geq s$ . In particular if  $r$  is the Gotzmann number of  $J$  we obtain, since  $I_r$  and  $J_r$  then have the same dimension, that the Plücker coordinate of  $I_r$  with respect to  $J_r$  is nonzero.

Then  $I$  is in the subscheme  $H_J$  of the Hilbert scheme defined by the nonvanishing of the Plücker coordinates with respect to  $J_r$ . By [3, Theorem 3.10]  $H_J$  is isomorphic to the *marked scheme*  $\mathcal{M}f(J_{\geq r})$  and again this is isomorphic to the marked scheme  $\mathcal{M}f(J_{\geq t})$  by [3, Theorem 4.4 iii)]. But then the Plücker coordinate of  $(I^{sat})_t$  with respect to  $J_t$  is nonzero and so  $\text{in}(I^{sat})$  has saturation equal to  $J$  by part a.

Now we show that if the projection of  $I_s$  onto the space of monomials in  $J_s$  is surjective, then the projection  $p$  of  $I_{s+1}$  onto the space of monomials in  $J_{s+1}$  is surjective. This is because there are filtrations

$$\begin{aligned} I_s \cdot (x_n) &\subseteq I_s \cdot (x_n, x_{n-1}) \subseteq \cdots \subseteq I_s \cdot (x_n, x_{n-1}, \dots, x_0) \subseteq I_{s+1} \\ J_s \cdot (x_n) &\subseteq J_s \cdot (x_n, x_{n-1}) \subseteq \cdots \subseteq J_s \cdot (x_n, x_{n-1}, \dots, x_0) = J_{s+1}. \end{aligned}$$

The projection  $I_{s+1} \rightarrow J_{s+1}$  respects this filtration. To see this, note that if  $x^a + \sum_j c_j x^{\mathbf{b}_j}$  in  $I_s$  maps to  $x^a$  in  $J_s$ , so the terms in the sum are not in  $J_s$ , then  $f = x_i x^a + \sum_j c_j x_i x^{\mathbf{b}_j}$  in  $I_s(x_n, \dots, x_i)$  maps to  $p(f) = x_i x^a + \sum_{x_i x^{\mathbf{b}_j} \in J_{s+1}} c_j x_i x^{\mathbf{b}_j}$  in  $J_{s+1}$ . But if  $x_i x^{\mathbf{b}_j}$  is in  $J_{s+1}$  then since  $J$  is Borel,  $x_i x^{\mathbf{b}_j} = x_{i'} x^{\mathbf{b}'_j}$  where  $x^{\mathbf{b}'_j} \in J_s$  and  $i' > i$ . Therefore the terms after the sigma in  $p(f)$  are in  $J_s(x_n, \dots, x_{i+1})$ . We also see by this argument that the induced maps

$$\frac{I_s(x_n, \dots, x_i)}{I_s(x_n, \dots, x_{i+1})} \xrightarrow{p_i} \frac{J_s(x_n, \dots, x_i)}{J_s(x_n, \dots, x_{i+1})}$$

are surjective. Hence the projection  $p$  is surjective.  $\square$

*Example 2.6.* The case  $l = 3$  and  $m = 1$ . This gives the Hilbert polynomial  $p(t) = 7t - 5$ . Using the applet BorelGenerator in [9], there are 112 saturated Borel ideals with this Hilbert polynomial. Of them, 18 fulfill conditions C1. and C2. The labels of these 18 given by BorelGenerator are

(5) 78, 79, 80, 82, 83, 85, 86, 95, 97, 99, 101, 102, 104, 105, 109, 110, 112.

Most of these are Hilbert segment ideals, or at least reg-segment ideals. For instance  $J_{79}$  is not a segment in degree  $r = 16$  (the Gotzmann number of  $p(t) = 7t - 5$ ), but it is a segment in degree  $m_0 = 9$  which is its regularity. One may check that the Plücker coordinate of the segment ideals are nonvanishing.

Only the following three cases are not segment ideals, as may be verified by the simple criterion [4, Prop. 3.5].

- $J_{83} = (x^2, xy^3, xy^2z, xyz^2, xz^6, y^7),$
- $J_{85} = (x^2, xy^2, xyz^4, xz^5, y^7),$
- $J_{102} = (x^2, xy^3, xy^2z, xyz^2, y^6z, y^7),$

For instance in the case of  $J_{85}$  one sees that  $xyz^3 \notin J_{85}$ , while  $xy^2z, xz^5 \in J_{85}$ . Since  $(xyz^3)^2 = xy^2z \cdot xz^5$  this ideal cannot be a segment. That these three ideals really are on the component may be verified as follows:

- Let  $I_{83}$  be the ideal generated by the  $2 \times 2$ -minors of  $A(F)$  where

$$F = y^2z + wzy + 2yz^2 - w^2z + 4z^3.$$

The initial ideal of  $I_{83}$  with respect to the lex order is a monomial ideal whose saturation is  $J_{83}$ .

- Let  $I_{85}$  be the ideal generated by  $2 \times 2$ -minors of  $A(F)$  where

$$F = y^2z + wzy - 2w^2y + yz^2 - 9w^2z + 3z^3 + 6w^3.$$

The initial ideal of  $I_{85}$  with respect to the lex order is a monomial ideal whose saturation is  $J_{85}$ .

- Finally let  $I_{102}$  be the ideal generated by  $2 \times 2$ -minors of  $A(F)$  where

$$F = y^2z + yz^2 + zw^2.$$

The initial ideal of  $I_{102}$  with respect to the monomial order  $\prec_{[10,3,2,1]}$  is an ideal whose saturation is  $J_{102}$ .

In conclusion all the ideals in the list (5) are on the component  $H(3, 1)$ .

The above three last cases would not work if we had chosen  $F$  to be general, instead we had to use special choices for  $F$ . The way to do this is explained in the next subsection.

**2.4. Ideals specializing to non-segment ideals.** To construct ideals like  $I_{83}$ ,  $I_{85}$  and  $I_{102}$  in the example above, we let

$$F = \sum_{i=0}^{l-1} y^{l-1-i} F_{m+i}$$

where

$$F_{m+i} = \sum C_{i,\alpha_1,\alpha_2} z^{\alpha_1} w^{\alpha_2}$$

is the general form in  $z$  and  $w$  of degree  $m+i$ , and the  $C_{i,\alpha}$  are variables. Denote by  $L$  the list of the three  $2 \times 2$ -minors of the matrix  $A(F)$  and  $I$  the ideal generated by these minors. Now fix a term order  $\prec$ , usually the lexicographic order, and let  $J$  be a Borel ideal. We want to assign values to the  $C_{i,\alpha}$  such that the initial ideal of  $I$  with respect to  $\prec$  is an ideal  $\hat{J}$  whose saturation is  $J$ . We apply a Buchberger-like algorithm as follows. We compute the  $S$ -polynomial of elements in  $L$  and reduce to

a polynomial  $h$ . Let  $m$  be the leading term in  $h$  and  $q(c)$  its coefficient, a polynomial in the  $C$ -variables.

1. If  $m$  is in  $J$  we add a variable  $c_m$  and a relation  $c_m q(c) - 1 = 0$ .
2. If  $m$  is not in  $J$  we add the relation  $q(c) = 0$ , where  $q(c)$  is the coefficient of  $m$ . Find the largest term in  $h$  after  $m$ . Let this be the new value of  $m$ . Continue with 2. until  $m$  is in  $J$ , and then go to 1.
3. Let  $L := L \cup \{h\}$ , and continue with 1. or 2. after computing a new  $S$ -polynomial.

In the end we get a system of equations in the  $C$ -variables. If we can find a solution to these we get an ideal  $I$  whose initial ideal will have  $J$  as its saturation. If the system has no solutions, we try again fixing a new term order, chosen so that 1. is used more often than 2. This procedure may not always succeed but in all cases we have used it, it does.

This was the procedure that enabled us to produce  $I_{83}$ ,  $I_{85}$  and  $I_{102}$  in Example 2.6 and most of the explicit Gröbner deformations in next example.

*Example 2.7.* The case  $l = 3$  and  $m = 3$ . The Hilbert polynomial is  $9t - 12$ . By the applet BorelGenerator there are 989 Borel ideals on  $\mathcal{Hilb}_{9t-12}^3$  and  $J_{989} = (x^2, xy^3, y^6)$  is the ACM Borel ideal. Among the remaining 988 ideals, only 45 fulfill the condition C1, and all of these, save  $J_{834} = (x^2, xy^3, xy^2z^4, xyz^5, xz^6, y^9)$ , fulfill the condition C2. There are 28 of these 44 ideals which are segments with respect to some term order and one may check that a general  $F$  gives Plücker coordinates which are nonzero. They are: 768, 769, 770, 772, 775, 780, 788, 801, 817, 875, 877, 880, 887, 898, 913, 927, 928, 938, 939, 941, 954, 955, 957, 960, 977, 979, 981, 984.

There are 16 ideals left to check. By using the procedure above we have been able to verify that all these ideals are on the component  $H(3, 3)$ . The first 10 can be obtained using the lexicographic term order.

- $J_{773} = (x^2, xy^3, xy^2z, xyz^2, y^9, xz^{12}), F = y^2z^3 - w^3z^2 + z^5 + 2wz^3y + w^2zy^2$
- $J_{776} = (x^2, xy^3, xy^2z, xyz^3, y^9, xz^{11}), F = y^2z^3 + z^3w^2 + 5z^4y + 25z^5 + 2z^3wy - 6w^2zy^2$
- $J_{781} = (x^2, xy^3, xy^2z, xyz^4, y^9, xz^{10}), F = y^2z^3 - wz^4 - z^5 - yz^4 - w^2zy^2$
- $J_{783} = (x^2, xy^3, xy^2z^2, xyz^3, y^9, xz^{10}), F = y^2z^3 + 4yz^4 + 11wz^4 - \frac{1127}{64}w^3z^2 + 3z^5 + 7wz^3y$
- $J_{787} = (x^2, xy^2, xyz^6, y^9, xz^9), F = y^2z^3 - 2wz^3y + 5w^2z^2y + z^5 - \frac{85}{9}w^2zy^2 + \frac{25}{3}w^3y^2 + 3wz^4$
- $J_{790} = (x^2, xy^3, xy^2z^2, xyz^4, y^9, xz^9), F = y^2z^3 + 2wz^3y - z^5 + 2wz^2y^2 - 2yz^4$

- $J_{798} = (x^2, xy^2, y^9, xyz^7, xz^8), F = 9y^2z^3 - 18wz^3y + 45w^2z^2y + 9z^5 - 85w^2zy^2 + 75w^3y^2 - 11yz^4 + 27wz^4$
- $J_{799} = (x^2, xy^3, xy^2z, xyz^6, y^9, xz^8), F = y^2z^3 - wz^2y^2 + 2wz^3y\sqrt{7} + 4w^2zy^2 + 7z^5$
- $J_{804} = (x^2, xy^3, xy^2z^3, xyz^4, y^9, xz^8), F = y^2z^3 + 8z^5 - 3wz^3y + 2wz^4 - 12yz^4$
- $J_{814} = (x^2, xy^3, xy^2z^2, xyz^6, xz^7, y^9), F = y^2z^3 + 6z^5 + 6wz^2y^2 + 6wz^3y + 6wz^4 + 6yz^4$

For the last six we use a different term order given by a matrix of the type:

$$M(v) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ v_1 & v_2 & v_3 & v_4 \\ v_5 & v_6 & v_7 & v_8 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

- $J_{888} = (x^2, xy^3, xy^2z, xy^2z^2, y^9, y^8z^5), F = y^2z^3 + wz^4 - 2z^5 + w^2zy^2 + w^3z^2 + w^2z^2y, v = [14, 2, 0, 0], [0, 0, 2, 1]$
- $J_{899} = (x^2, xy^3, xy^2z, xy^2z^3, y^9, y^8z^4), F = y^2z^3 - 2z^5 + w^2zy^2 + w^2z^2y + wz^4, v = [14, 2, 0, 0], [0, 0, 2, 1]$
- $J_{914} = (x^2, xy^3, xy^2z, xyz^4, y^9, y^8z^3), F = y^2z^3 - 2z^5 + w^2zy^2 - yz^4 + wz^4, v = [14, 2, 0, 0], [0, 0, 2, 1]$
- $J_{930} = (x^2, xy^3, xy^2z^2, xyz^4, y^9, y^8z^2), F = y^2z^3 - 2z^5 + wz^4 - w^2z^2y, v = [14, 2, 0, 0], [0, 0, 1, -1]$
- $J_{944} = (x^2, xy^3, xy^2z^3, xyz^4, y^9, y^8z), F = y^2z^3 + yz^4 - 2z^5 + wz^4, v = [14, 2, 0, 0], [0, 0, 7, 1]$
- $J_{978} = (x^2, xy^2, y^9, y^8z, y^7z^2), F = y^2z^3 + w^2z^2y - w^3zy + wz^2y^2 + w^4z - 3w^5, v = [12, 2, 0, 0], [0, 0, 7, 1]$

### 3. BOREL IDEALS ON COMPONENTS OF ACM CURVES

We now consider ACM curves in  $\mathbb{P}^3$  and investigate Borel ideals on the Hilbert scheme component of such curves. We give various sufficient conditions for a Borel ideal to be on such a component.

The idea is to construct families of ACM curves and find Borel ideals which are specializations of such families. This is intricate and we shall focus here on the case that the curves are on a quadric, that is the case  $a = 2$  in (1). Related to this is [10] by P.Lella where he constructs families of ideal parametrized by a rational curve, connecting two Borel ideals. In this way he can proceed stepwise between Borel ideals. However as soon as one has used two steps or more, one cannot be sure that the starting and ending Borel ideal is on the same component.

**3.1. Conditions on Borel ideals.** Let  $J$  be a saturated Borel ideal on the component  $H(l, m)$  of  $J(l, m) = (x^2, xy^l, y^{l+m})$ . By Theorem 1.5 all the monomials  $x^2, xy^l$  and  $y^{2l+m}$  are in the ideal  $J$ . Hence  $J$



must have the following set of generators for some  $0 \leq p \leq l$ .

$$(6) \quad \begin{aligned} & x^2, \\ & xy^{l-p}z^{a_0}, xy^{l-p+1}z^{a_1}, \dots, xy^{l-1}z^{a_{p-1}}, xy^l, \\ & y^{l+m+p}z^{b_p}, \dots, y^{2l+m-1}z^{b_{l-1}}, y^{2l+m}, \end{aligned}$$

where the sequences  $a_0, a_1, \dots$  and  $b_p, b_{p+1}, \dots$  are strictly decreasing until they possibly become zero. It will also be convenient to allow these sequences to be weakly decreasing. We call such an ideal an *almost Borel ideal*.

**Lemma 3.1.** *The ideal  $J(l, m)$  and the ideal  $J$  with generators (6) have the same Hilbert polynomial iff  $\sum a_i + \sum b_i = \sum_{i=0}^{p-1} (m + 2i)$ .*

*Proof.* Suppose  $a_j > a_{j+1}$ . Let  $J'$  be the ideal with the generator  $xy^{l-j}z^{a_j}$  replaced by  $xy^{l-j}z^{a_j-1}$ . There is then an exact sequence

$$0 \rightarrow (xy^{l-p+j}z^{a_j-1}) \rightarrow S/J \rightarrow S/J' \rightarrow 0$$

where the first submodule consists of elements  $xy^{l-p+j}z^{a_j-1}w^r$  for  $r \geq 0$ . Hence the difference of the Hilbert polynomials of the two latter quotient rings is just 1. The same thing happens when we reduce some  $b_k$ .

Now let  $J_{p,q}$  be the ideal generated by  $x^2, xy^{l-p}, y^{l+m+q}$ . By successively reducing the  $a$ 's and  $b$ 's we find for the Hilbert polynomials

$$(7) \quad HP_{J_{p,p}}(d) = HP_J(d) + \sum a_i + \sum b_i.$$

Now there is an exact sequence

$$0 \rightarrow (xy^{l-p-1}) \rightarrow S/J_{p,q} \rightarrow S/J_{p+1,q} \rightarrow 0$$

and the kernel is  $xy^{l-p-1}\langle z, w \rangle^{d-(l-p)}$  in degree  $d$  and so has Hilbert polynomial  $d - (l - p) + 1$ . There is also an exact sequence

$$0 \rightarrow (y^{l+m+q}) \rightarrow S/J_{p,q+1} \rightarrow S/J_{p,q} \rightarrow 0.$$

The kernel has Hilbert polynomial  $d - (l + m + q) + 1$ . From this we readily get that the Hilbert polynomial  $HP_{J_{p+1,p+1}}(d)$  is equal to  $HP_{J_{p,p}}(d) + (m + 2p)$ . Starting with  $J_{0,0} = J(l, m)$  we therefore get that

$$HP_{J_{p,p}}(d) = HP_{J(l,m)}(d) + \sum_{i=0}^{p-1} (m + 2i).$$

Comparing this with (7),  $J(l, m)$  and  $J$  have the same Hilbert function iff the numerical equality holds.  $\square$



**3.2. Main theorems of sufficiency.** For Borel ideals with  $p \leq 3$  we now give sufficient conditions for a Borel ideal to be on the component  $H(l, m)$ .

**Theorem 3.2** ( $p = 1$ ). *When  $\mathbf{a} = (m - i)$  and  $\mathbf{b} = (i, 0, \dots, 0)$ , the Borel ideal (6) is on the component  $H(l, m)$ .*

**Theorem 3.3** ( $p = 2$ ). *When  $\mathbf{a} = (a_0, a_1)$  and  $\mathbf{b} = (b_2, 0, \dots, 0)$  with  $a_0 + a_1 + b_2 = 2m + 2$ , the Borel ideal (6) is on the component  $H(l, m)$  if either:*

1.  $a_0 \geq m + 2$ , or
2.  $a_0 \leq m + 2$  and  $a_0 - a_1$  is even.

**Theorem 3.4** ( $p = 3$ ). *When  $\mathbf{a} = (a_0, a_1, a_2)$  and  $\mathbf{b} = \mathbf{0}$  with  $a_0 + a_1 + a_2 = 3m + 6$  the Borel ideal (6) is on the component  $H(l, m)$  if one of the following holds*

1.  $a_1 < m + 2$  except possibly if  $a_2 = a_1 - 1$  and  $3a_1 \leq 2m + 4$ .
2.  $a_1 = m + 2$  and  $a_2 \leq m$ .
3.  $\mathbf{a} = (m + 2 + e + a, m + 2 + e, m + 2 - 2e - a)$  where  $e, a > 0$ ,  $m + 2 \geq 2e + a$  and  $a$  has an odd prime factor and for the smallest such factor  $p$ , we have  $pe \leq a$ .

The proofs of these theorems will be at the end of Section 5.

**Discussion of case 3.** For low values of  $e$  we get more informative descriptions as follows.

1. When  $e = 1$ , the condition is that  $\mathbf{a} = (m + 3 + a, m + 3, m - a)$  where  $a \leq m$  is a positive integer not a power of 2.
2. When  $e = 2$ , the condition is that  $\mathbf{a} = (m + 4 + a, m + 4, m - 2 - a)$  where  $a \leq m - 2$  is a positive integer not
  - a prime number,
  - a power of 2.
3. When  $e = 3$ , the condition is that  $\mathbf{a} = (m + 5 + a, m + 5, m - 4 - a)$  where  $a \leq m - 4$  is a positive integer not
  - a prime number,
  - twice a prime number,
  - a power of 2.

We also have a more general result. Given non-negative integers with

$$p_0 \leq p_1 + 1 \leq p_2 + 2 \leq \dots \leq p_{l-1} + l - 1.$$

Consider partitions

$$\lambda : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r (\geq 0)$$

consisting of  $r$  parts of sizes  $\leq l - 1$ . Let  $p_\lambda = \sum p_{\lambda_i}$ .

**Theorem 3.5.** *Assume for each  $r = 0, \dots, l - 1$  that  $rp_{l-r} \geq p_\lambda$  for all partitions  $\lambda$  of  $r(l - r)$  into  $r$  parts of sizes  $\leq l - 1$ . (In other words  $p_\lambda$*

achieves its maximum when all parts are equal.) The Borel ideal (6) with  $p = l$  and

$$a_i = m + 2(l - 1 - i) + (l - 1 - i)p_{i+1} - (l - i)p_i + p_0$$

for  $i = 0, \dots, l - 1$ , is on the component  $H(l, m)$ .

In particular letting each  $p_i = 0$  we see that the ideal with  $a_i = m + 2(l - 1 - i)$  for  $i = 0, \dots, l - 1$  is on the component  $H(l, m)$ .

**3.3. Auxiliary results.** In the end we now give some auxiliary results which will be repeatedly used in our arguments for the above results. When  $I$  is a monomial ideal in  $k[x, y, z, w]$  we may make a coordinate change  $w \rightarrow w + \lambda z$  and let  $J$  be the initial ideal for any monomial order where  $z > w$ . Note that if  $x^a y^b z^{c_1} w^{c_2}$  is in  $I$ , then  $x^a y^b z^{c_1+c_2}$  is in  $J$ . The following is clear.

**Lemma 3.6.** *Any component of the Hilbert scheme containing  $I$  will also contain  $J$ .*

We call  $J$  the  $z$ -transform of  $I$ . The following will be used frequently.

**Saturation Lemma 3.7.** *Let  $I$  be a monomial ideal with the same Hilbert polynomial as  $J(l, m)$ . If the saturation of  $I$  (resp. the  $z$ -transform of  $I$ ) contains an almost Borel ideal  $K$  of the form (6) with*

$$\sum a_i + \sum b_i = \sum_{i=0}^{p-1} (m + 2i),$$

*then  $K$  is the saturation of  $I$  (resp. the  $z$ -transform of  $I$ ).*

*Proof.* Clearly  $I$  and  $K$  have the same Hilbert polynomial. So  $K/I$  is of finite length. Since  $K$  is saturated, it must be the saturation of  $I$ . The argument for the  $z$ -transform is similar.  $\square$

#### 4. EQUATIONS OF FAMILIES OF ACM CURVES ON A QUADRIC

We will now describe explicitly the families of ACM curves that we shall work with and whose degenerations will be Borel ideals.

**4.1. The family of ideals.** Denote by  $R$  a polynomial ring  $k[s, \{t_k\}]$ . This will be a parameter space for the family of ideals  $\tilde{I}$  we shall work with. This is the family generated by the  $2 \times 2$  minors of the matrix

$$A_R(F) = \begin{bmatrix} x & sy^m & -F \\ 0 & x & y^l \end{bmatrix}$$

where  $F$  is a polynomial in  $R[x, y, z, w]$ , homogeneous of degree  $l+m-1$  in the  $x, y, z, w$ . The  $2 \times 2$  minors of the matrix are

$$x^2, \quad xy^l, \quad G := xF + sy^{m+l}.$$

Performing row and column operations on the matrix, we may assume

$$F = y^{l-1}F_{l-1} + y^{l-2}F_{l-2} + \cdots + y^{l-q}F_{l-q}.$$

Of course the most general is having  $q = l$ . We write it however in this way since  $q$  will be a natural parameter in the families we construct.

We now assume that each  $F_{l-i}$  is the product of a monomial in  $z, w$  and a variable in the ring  $R$ . More specifically we assume that  $F$  has the following form

$$t_0 y^{l-1} z^{m-p_{l-1}} w^{p_{l-1}} + t_1 y^{l-2} z^{m+1-p_{l-2}} w^{p_{l-2}} + \cdots + t_{l-1} z^{m+l-1-p_0} w^{p_0}.$$

Then  $\tilde{I}$  is generated by  $x^2$ ,  $xy^l$ , and

$$G_0 = G = t_0 xy^{l-1} z^{m-p_{l-1}} w^{p_{l-1}} + \cdots + t_{l-1} x z^{m+l-1-p_0} w^{p_0} + sy^{m+l}.$$

Via a ring homomorphism  $R = k[s, t_0, \dots, t_{l-1}] \rightarrow k[t, t^{-1}]$ , the family of ideals  $\tilde{I}$  maps to a family of ideals  $\tilde{I}_t$  parametrized by a rational parameter  $t$ . This new family has a limit ideal when  $t \rightarrow \infty$ . We shall consider monomial maps where  $s \mapsto t^w$  and  $t_i \mapsto t^{w_i}$ . When  $w$  and the  $w_i$ 's are sufficiently general this limit will be a monomial ideal.

Now let  $R$  have a monomial order. Given an integer  $N$ , by D.Bayer's thesis, [1], there are integer weights  $w$  and  $w_i$ 's such that for two monomials of degrees  $\leq N$ ,

$$m_{\mathbf{a}} = s^a t_0^{a_i} \cdots t_{l-1}^{a_{l-1}}, \quad m_{\mathbf{b}} = s^b t_0^{b_i} \cdots t_{l-1}^{b_{l-1}},$$

we have  $m_{\mathbf{a}} > m_{\mathbf{b}}$  iff the scalar products  $wa + \sum w_i a_i > wb + \sum w_i b_i$ . For this set of weights associate the map  $R \rightarrow k[t, t^{-1}]$  given by  $s \mapsto t^w$  and  $t_i \mapsto t^{w_i}$ . We then get a monomial ideal as the limit of  $\tilde{I}_t$  when  $t \rightarrow \infty$ . This limit ideal depends only on the family  $\tilde{I}$  and the monomial ordering on  $R$  and not on the choice of weights  $w$  and  $w_i$ 's.

We shall in the next sections consider various monomial orders on  $R$  and find the limit ideals associated to these orders.

The ideal  $\tilde{I}$  contains  $yG_0$ . Since  $xy^l$  is in  $\tilde{I}$ , this is congruent modulo  $\tilde{I}$  to

$$G_1 = t_1 xy^{l-1} z^{m+1-p_{l-2}} w^{p_{l-2}} + \cdots + t_{l-1} xy z^{m+l-1-p_0} w^{p_0} + sy^{m+l+1}.$$

More generally  $y^r G_0$  is congruent modulo  $\tilde{I}$  to

$$G_r = t_r xy^{l-1} z^{m+r-p_{l-1-r}} w^{p_{l-1-r}} + \cdots + t_{l-1} xy^r z^{m+l-1-p_0} w^{p_0} + sy^{m+l+r}.$$

Note that all the polynomials  $G_0, G_1, \dots, G_{l-1}$  are in  $\tilde{I}$ .

**4.2. Equations and limits when  $q = 2$ .** We let  $q = 2$ ,  $p_1 = i$  and  $p_0 = 0$ . In this case we get

$$\begin{aligned} G_0 &= t_0 xy^{l-1} z^{m-i} w^i + t_1 xy^{l-2} z^{m+1} + sy^{l+m} \\ G_1 &= t_1 xy^{l-1} z^{m+1} + sy^{l+m+1}. \end{aligned}$$

In order to eliminate  $xy^{l-1}$  from these equations we let

$$\begin{aligned} G_{01} &= t_1 z^{i+1} G_0 - t_0 w^i G_1 \\ &= t_1^2 xy^{l-2} z^{m+2+i} + st_1 y^{m+2} z^{i+1} - st_0 y^{l+m+1} w^i. \end{aligned}$$

**Proposition 4.1.** *Let  $t_0 > t_1 > s$ .*

1. *When  $t_1^2 > st_0$  the saturation of the limit ideal is Borel with  $\mathbf{a} = (m+2+i, m-i)$  and  $\mathbf{b} = \mathbf{0}$ .*
2. *When  $st_0 > t_1^2$  the saturation of the  $z$ -transform of the limit ideal is Borel with  $\mathbf{a} = (m-i)$  and  $\mathbf{b} = (i, 0, \dots, 0)$ .*

As a consequence of 2. we get Theorem 3.2.

*Proof.* By  $G_1$  the limit contains  $xy^{l-1}z^{m+1}$ , and by  $G_0$  the limit contains  $xy^{l-1}z^{m-i}w^i$ . Hence the saturation of the limit contains  $xy^{l-1}z^{m-i}$ .

In case 1. the limit contains  $xy^{l-2}z^{m+2+i}$  by  $G_{01}$ , proving a. by the Saturation Lemma 3.7. In case 2. the limit contains  $y^{l+m+1}w^i$ . Again by the Saturation Lemma 3.7 the  $z$ -transform of the limit ideal has the type given.  $\square$

**4.3. The equations when  $q = 3$ .** We let  $p_2 = i$ ,  $p_1 = j$ , and  $p_0 = 0$ . Also let  $\Delta = i - 2j$ , the second difference of the  $p$ 's. The ideal  $\tilde{I}$  contains:

$$\begin{aligned} G_0 &= t_0 xy^{l-1} z^{m-i} w^i + t_1 xy^{l-2} z^{m+1-j} w^j + t_2 xy^{l-3} z^{m+2} + sy^{l+m} \\ G_1 &= t_1 xy^{l-1} z^{m+1-j} w^j + t_2 xy^{l-2} z^{m+2} + sy^{l+m+1} \\ G_2 &= t_2 xy^{l-1} z^{m+2} + sy^{l+m+2}. \end{aligned}$$

We now eliminate  $xy^{l-1}$  from  $G_0$  and  $G_1$ . When  $i \geq j$  we let

$$\begin{aligned} G_{01} &= t_1 z^{1+i-j} G_0 - t_0 w^{i-j} G_1 \\ &= t_1^2 xy^{l-2} z^{m+2+\Delta} w^j + t_1 t_2 xy^{l-3} z^{m+3+\Delta+j} + st_1 y^{l+m} z^{1+\Delta+j} \\ &\quad - t_0 t_2 xy^{l-2} z^{m+2} w^{\Delta+j} - st_0 y^{l+m+1} w^{\Delta+j}. \end{aligned}$$

We eliminate  $xy^{l-1}$  from  $G_1$  and  $G_2$  by letting

$$\begin{aligned} G_{12} &= t_2 z^{1+j} G_1 - t_1 w^j G_2 \\ &= t_2^2 xy^{l-2} z^{m+3+j} + st_2 y^{l+m+1} z^{1+j} - st_1 y^{l+m+2} w^j. \end{aligned}$$

Now we want to eliminate  $xy^{l-1}$  and  $xy^{l-2}$  from the equations of  $G_0, G_1$ , and  $G_2$ . By taking  $2 \times 2$ -minors of the coefficients of these monomials we let

$$G_{012} = t_2^2 z^{2m+4} G_0 - t_1 t_2 z^{2m+3-j} w^j G_1 + (t_1^2 z^{2m+2-2j} w^{2j} - t_0 t_2 z^{2m+2-i} w^i) G_2.$$

When  $\Delta \geq 0$ , which is equivalent to  $i \geq 2j$  we can factor out  $z^{2m+2-i}$  from the coefficients and let

$$\begin{aligned} G_{012}^+ &= G_{012} / z^{2m+2-i} \\ &= t_2^3 xy^{l-3} z^{m+4+i} + st_2^2 y^{l+m} z^{i+2} - st_1 t_2 y^{l+m+1} z^{1+i-j} w^j \\ &\quad + st_1^2 y^{l+m+2} z^{i-2j} w^{2j} - st_0 t_2 y^{l+m+2} w^i. \end{aligned}$$

When  $\Delta \leq 0$ , which is equivalent to  $i \leq 2j$  we can factor out  $z^{2m+2-2j}$  from the coefficients and let

$$\begin{aligned} G_{012}^- &= G_{012}/z^{2m+2-2j} \\ &= t_2^3 xy^{l-3} z^{m+4+2j} + st_2^2 y^{l+m} z^{2j+2} - st_1 t_2 y^{l+m+1} z^{1+j} w^j \\ &\quad + st_1^2 y^{l+m+2} w^{2j} - st_0 t_2 y^{l+m+2} z^{2j-i} w^i. \end{aligned}$$

The equations  $G_{012}^+$  and  $G_{012}^-$  give respectively

$$(8) \quad xy^{l-3} z^{m+4+i} \equiv 0 \pmod{\tilde{I}, s}$$

$$(9) \quad xy^{l-3} z^{m+4+2j} \equiv 0 \pmod{\tilde{I}, s}.$$

We get from the expression of  $G_{01}$  that when  $i \geq j$

$$(10) \quad xy^{l-2} z^{m+2} w^{j+\Delta} \equiv \frac{t_1^2}{t_0 t_2} xy^{l-2} z^{m+2+\Delta} w^j + \frac{t_1}{t_0} xy^{l-3} z^{m+3+j+\Delta} \pmod{\tilde{I}, s}.$$

By multiplying (10) with  $z^\Delta$  we get at the left side  $xy^{l-2} z^{m+2+\Delta} w^{j+\Delta}$ . The expression for this may be inserted in the first term on the right side of (10) after multiplying with  $w^\Delta$ . We then get

$$\begin{aligned} xy^{l-2} z^{m+2} w^{j+2\Delta} &\equiv \frac{t_1^4}{t_0^2 t_2^2} xy^{l-2} z^{m+2+2\Delta} w^j + \frac{t_1}{t_0} xy^{l-3} z^{m+3+j+\Delta} w^\Delta \\ &\quad + \frac{t_1^3}{t_0^2 t_2} xy^{l-3} z^{m+3+j+2\Delta} \pmod{\tilde{I}, s}. \end{aligned}$$

Iterating this insertion we get modulo  $\tilde{I}$  and  $s$  when  $\Delta \geq 0$ :

$$(11) \quad xy^{l-2} z^{m+2} w^{j+r\Delta} \equiv \frac{t_1^{2r}}{t_0^r t_2^r} xy^{l-2} z^{m+2+r\Delta} w^j + \sum_{k=1}^r \frac{t_1^{2k-1}}{t_0^k t_2^{k-1}} xy^{l-3} z^{m+3+j+k\Delta} w^{(r-k)\Delta}.$$

Let  $\Delta^- = -\Delta$  we get from the expression of  $G_{01}$  that when  $i \geq j$

$$(12) \quad xy^{l-2} z^{m+2-\Delta^-} w^j \equiv \frac{t_0 t_2}{t_1^2} xy^{l-2} z^{m+2} w^{j-\Delta^-} - \frac{t_2}{t_1} xy^{l-3} z^{m+3+j-\Delta^-} \pmod{\tilde{I}, s}.$$

Since (12) is the analog of (10), when  $\Delta \leq 0$  the analog of (11) is

$$(13) \quad \begin{aligned} xy^{l-2} z^{m+2-\Delta^-} w^{j+(r-1)\Delta^-} &\equiv \frac{t_0^r t_2^r}{t_1^{2r}} xy^{l-2} z^{m+2+(r-1)\Delta^-} w^{j-\Delta^-} \\ &\quad - \sum_{k=1}^r \frac{t_2^k t_0^{k-1}}{t_1^{2k-1}} xy^{l-3} z^{m+3+j+(k-2)\Delta^-} w^{(r-k)\Delta^-}. \end{aligned}$$

Finally we get by  $G_{12}$  that

$$(14) \quad xy^{l-2} z^{m+3+j} \equiv 0 \pmod{\tilde{I}, s}.$$

5. THE LIMIT IDEALS WHEN  $q = 3$ 

We go systematically through the different monomial orders in the ring  $R$  and compute the limit ideals.

In the first subsections we assume we have an order where monomials containing  $s$  are smaller than those not containing  $s$ . The saturation of the limit or its  $z$ -transform will then always be an ideal with  $p = 3$ . Only in the last subsection do we consider the case of arbitrary monomial orders. For simplicity we also assume that  $l = 3$ . We do not lose any generality by this, since we may simply multiply all equations (save  $x^2$ ) by  $y^{l-3}$  to get the case of general  $l$ .

Let us make a first observation.

**Basic Limits Lemma 5.1.** *Assume all monomials in  $R$  containing  $s$  are smaller than those not containing  $s$ .*

- a.  $xz^{m+4+i}$  is in the limit ideal when  $\Delta \geq 0$ .
- b.  $xz^{m+4+2j}$  is in the limit when  $\Delta \leq 0$ .
- c.  $xyz^{m+2+\Delta}$  is in the saturation of the limit when  $i \geq j$ ,  $t_1 > t_2$  and  $t_1^2 > t_0 t_2$ .
- d.  $xyz^{m+2}$  is in the saturation of the limit when  $i \geq j$ ,  $t_0 > t_1$  and  $t_0 t_2 > t_1^2$ .
- e.  $xy^2 z^{m-i}$  is in the saturation of the limit when  $t_0$  is bigger than  $t_1$  and  $t_2$ .

*Proof.* Parts a. and b. follows by equations (8) and (9). In case e., by  $G_0$  the limit contains  $xy^2 z^{m-i} w^i$  and by  $G_2$  it contains  $xy^2 z^{m+2}$ . So the saturation contains the stated monomial.

To see d. note that by (10)  $xyz^{m+2} w^{j+\Delta}$  is in the limit. By a. and b.  $xyz^N$  is in the limit for large  $N$ . Hence the saturation contains  $xyz^{m+2}$ . Part c. follows similarly by considering (12).  $\square$

**5.1. The case  $t_0 > t_1 > t_2$  and  $i \geq j$ .** In this subsection we assume these order properties.

**Corollary 5.2.** a. *Assume  $\Delta \leq 0$  and  $t_1^2 > t_0 t_2$ . The saturation of the limit ideal is the Borel ideal with  $\mathbf{a} = (m + 4 + 2j, m + 2 + i - 2j, m - i)$ .*  
 b. *Assume  $\Delta \geq 0$  and  $t_0 t_2 > t_1^2$ . The saturation of the limit ideal is the Borel ideal with  $\mathbf{a} = (m + 4 + i, m + 2, m - i)$ .*

*Proof.* This follows by the above Basic Limits Lemma and by the Saturation Lemma 3.7.  $\square$

**Proposition 5.3.** *Assume  $\Delta \geq 0$  and  $t_1^2 > t_0 t_2$ . Let  $\rho$  be maximal such that  $\rho \Delta \leq j + 1$  and let  $\rho \Delta + \Delta_2 = j + 1$  where  $0 \leq \Delta_2 < \Delta$ . The saturation of the limit ideal is the Borel ideal with*

- a.  $\mathbf{a} = (m + 4 + 2j, m + 2 + i - 2j, m - i)$  if  $t_1^{2\rho-1} > t_0^\rho t_2^{\rho-1}$ .
- b.  $\mathbf{a} = (m + 4 + 2j + \Delta - \Delta_2, m + 2 + \Delta_2, m - i)$  if  $t_1^{2\rho-1} < t_0^\rho t_2^{\rho-1}$  and  $t_1^{2\rho+1} > t_0^{\rho+1} t_2^\rho$ .

c.  $\mathbf{a} = (m + 4 + i, m + 2, m - i)$  if  $t_1^{2\rho+1} < t_0^{\rho+1}t_2^\rho$ .

*Proof.* The following monomials are in the saturation of the limit ideal by the Basic Limits Lemma 5.1: 1.  $xy^2z^{m-i}$ , 2.  $xyz^{m+2+i-2j}$ , and 3.  $xz^{m+4+i}$ . Consider (11) when  $r = \rho$ . Using, by (14), that  $xyz^{m+3+j} \equiv 0 \pmod{\tilde{I}, s}$ , we see by multiplying (11) with  $z^{\Delta_2}$ , that the first term on the right side of (11) will be  $\equiv 0 \pmod{\tilde{I}, s}$ .

Hence when  $t_1^{2\rho-1} < t_0^\rho t_2^{\rho-1}$  the saturation of the limit contains  $xyz^{m+2+\Delta_2}$  and when these  $t$ -monomials have the opposite order the saturation of the limit contains  $xz^{m+3+j+r\Delta+\Delta_2} = xz^{m+4+2j}$ . This settles case a. by the Saturation Lemma 3.7.

Now consider equation (11) with  $r = \rho + 1$ . Then the first term on the right side of this equation is  $\equiv 0 \pmod{\tilde{I}, s}$  by (14). Hence when  $t_1^{2\rho+1} < t_0^{\rho+1}t_2^\rho$ , the saturated limit contains  $xyz^{m+2}$  proving c. by the Saturation Lemma 3.7. When these  $t$ -monomials have the opposite order, the saturated limit contains  $xz^{m+3+j+(\rho+1)\Delta} = xz^{m+4+2j+\Delta-\Delta_2}$ . This settles case b. by the Saturation Lemma 3.7.  $\square$

**Proposition 5.4.** *Assume  $\Delta \leq 0$  and  $t_0t_2 > t_1^2$ . For convenience let  $\Delta^- = -\Delta$ . Let  $\rho$  be maximal such that  $\rho\Delta^- \leq j + 1$  and let  $\rho\Delta^- + \Delta_2^- = j + 1$  where  $0 \leq \Delta_2^- < \Delta^-$ . The saturation of the limit ideal is the Borel ideal with*

- a.  $\mathbf{a} = (m + 4 + i, m + 2, m - i)$  if  $t_2^{\rho+1}t_0^\rho > t_1^{2\rho+1}$ .
- b.  $\mathbf{a} = (m + 4 + 2j - \Delta_2^-, m + 2 - \Delta^- + \Delta_2^-, m - i)$  if  $t_2^{\rho+1}t_0^\rho < t_1^{2\rho+1}$  and  $t_2^{\rho+2}t_0^{\rho+1} > t_1^{2\rho+3}$ .
- c.  $\mathbf{a} = (m + 4 + 2j, m + 2 + i - 2j, m - i)$  if  $t_2^{\rho+2}t_0^{\rho+1} < t_1^{2\rho+3}$ .

*Proof.* The following monomials are in the saturation of the limit ideal by the Basic Limits Lemma 5.1: 1.  $xy^2z^{m-i}$ , 2.  $xyz^{m+2}$ , and 3.  $xz^{m+4+2j}$ . Now consider (13) with  $r = \rho + 1$ . Multiplying (13) with  $z^{\Delta_2^-}$  and using (14), we see that when  $t_2^{\rho+1}t_0^\rho < t_1^{2\rho+1}$  the saturated limit contains  $xyz^{m+2-\Delta^-+\Delta_2^-}$  and when these  $t$ -monomials have the opposite order the saturated limit contains  $xz^{m+3+j+(\rho-1)\Delta^-+\Delta_2^-}$  which is  $xz^{m+4+2j-\Delta^-} = xz^{m+4+i}$  proving a.

Now consider (13) with  $r = \rho + 2$ . The first term on the right of (13) is then  $\equiv 0 \pmod{\tilde{I}, s}$  by (14). So when  $t_2^{\rho+2}t_0^{\rho+1} < t_1^{2\rho+3}$  the saturated limit contains  $xyz^{m+2-\Delta^-} = xyz^{m+2+i-2j}$  which proves c. by the Saturation Lemma 3.7. When these  $t$ -monomials have the opposite order the saturated limit contains  $xz^{m+3+j+\rho\Delta^-} = xz^{m+4+2j-\Delta_2^-}$ . This settles b. by the Saturation Lemma 3.7.  $\square$

**5.2. The case  $t_0 > t_1 > t_2$  and  $i < j$ .** Under this assumption, to eliminate  $xy^2$  from  $G_0$  and  $G_1$  we must form

$$\begin{aligned} G'_{01} &= t_1 w^{j-i} G_0 - t_0 z^{j-i-1} G_1 \\ &= t_1^2 x y z^{m+1-j} w^{\Delta^-} + t_1 t_2 x z^{m+2} w^{\Delta^- - j} + s t_1 y^{m+3} w^{j-i} \\ &\quad - t_0 t_2 x y z^{m+1+\Delta^- - j} - s t_0 y^{m+4} z^{\Delta^- - j-1}. \end{aligned}$$

This gives

$$(15) \quad x y z^{m+1-j} w^{\Delta^-} \equiv \frac{t_0 t_2}{t_1^2} x y z^{m+1-j+\Delta^-} - \frac{t_2}{t_1} x z^{m+2} w^{\Delta^- - j} \pmod{\tilde{I}, s}.$$

**Proposition 5.5.** *Suppose  $i \leq j-1$ . The saturation of the limit ideal is the almost Borel ideal with*

- a.  $\mathbf{a} = (m+4+2j, m+1-j, m+1-j)$  if  $t_1^2 > t_0 t_2$ , or  $t_1^2 < t_0 t_2$  and  $t_1^3 > t_2^2 t_0$ .
- b.  $\mathbf{a} = (m+2+2j-i, m+3+i-j, m+1-j)$  if  $t_1^2 < t_0 t_2$  and  $t_1^3 < t_2^2 t_0$ .

*Proof.* By the Basic Limits Lemma 5.1 the limit contains  $x z^{m+4+2j}$ . Also by  $G_1$  the limit contains  $x y^2 z^{m+1-j}$ .

Case  $t_1^2 > t_0 t_2$ . By (15) the limit contains  $x y z^{m+1-j} w^{\Delta^-}$  and so the saturation contains  $x y z^{m+1-j}$ . By the Saturation Lemma 3.7 this settles

a. when  $t_1^2 > t_0 t_2$ .

Case  $t_1^2 < t_0 t_2$ . Multiplying (15) with  $z^{i+2}$  and using (14) saying that  $x y z^{m+3+j} \equiv 0 \pmod{\tilde{I}, s}$ , the saturated limit contains  $x y z^{m+3+i-j}$ . Iterating (15) once we obtain

$$\begin{aligned} x y z^{m+1-j} w^{2\Delta^-} &\equiv \frac{t_0^2 t_2^2}{t_1^4} x y z^{m+1-j+2\Delta^-} - \frac{t_2}{t_1} x z^{m+2} w^{2\Delta^- - j} \\ &\quad - \frac{t_2^2 t_0}{t_1^3} x z^{m+2+\Delta^-} w^{\Delta^- - j}. \end{aligned}$$

Now notice that  $m+1-j+2\Delta^- \geq m+3+j$  and so by (14) the first term on the right above is  $\equiv 0 \pmod{\tilde{I}, s}$ . When  $t_1^3 > t_2^2 t_0$  the saturation of the limit contains  $x y z^{m+1-j}$  giving a. When  $t_1^3 < t_2^2 t_0$  the saturation of the limit contains  $x z^{m+2+\Delta^-} = x z^{m+2+2j-i}$  which settles b. by the Saturation Lemma 3.7.  $\square$

**5.3. The case  $t_1$  largest.** Until now we have considered the case when  $t_0 > t_1 > t_2$  but now we consider the case when  $t_1$  is largest.

For this we need to eliminate  $xy$  from  $G_0$  and  $G_1$ . So let

$$\begin{aligned} G''_{01} &= t_2 z^{j+1} G_0 - t_1 w^j G_1 \\ &= t_0 t_2 x y^2 z^{m+1+j-i} w^i + t_2^2 x z^{m+j+3} + s t_2 y^{m+3} z^{j+1} \\ &\quad - t_1^2 x y^2 z^{m+1-j} w^{2j} - s t_1 y^{m+4} w^j, \end{aligned}$$



and so we get:

$$(16) \quad xy^2z^{m+1+j-i}w^i \equiv \frac{t_1^2}{t_0t_2}xy^2z^{m+1-j}w^{2j} - \frac{t_2}{t_0}xz^{m+j+3} \pmod{\tilde{I}, s}.$$

**Proposition 5.6.** *The saturation of the limit ideal is the almost Borel ideal with*

- a.  $\mathbf{a} = (m+4+2j, m+1-j, m+1-j)$  if  $i \leq 3j+1$ .
- b.  $\mathbf{a} = (m+3+i-j, m+1-j, m+2+2j-i)$  if  $i \geq 3j+2$  and  $t_0 > t_2$ .
- c.  $\mathbf{a} = (m+4+2j, m+1-j, m+1-j)$  if  $i \geq 3j+2$  and  $t_2 > t_0$ .

*Proof.* By the Basic Limits Lemma 5.1 either  $xz^{m+4+2j}$  (if  $\Delta \leq 0$ ) or  $xz^{m+4+i}$  ( $\Delta \geq 0$ ) is in the limit. Since  $t_1$  is now largest, by looking at  $G_0$  the limit contains  $xyz^{m+1-j}w^j$  and so the saturation  $xyz^{m+1-j}$ . Then  $xy^2z^{m+1-j}$  is also in this saturation. When  $i \leq 2j$  (i.e.  $\Delta \leq 0$ ) we get a. by the Saturation Lemma 3.7.

Now let  $i \geq 2j$ . Iterating (16) once we have

$$(17) \quad \begin{aligned} xy^2z^{m+1+j-i}w^{i+\Delta} &\equiv \frac{t_1^4}{t_0^2t_2^2}xy^2z^{m+1-j+\Delta}w^{2j} - \frac{t_2}{t_0}xz^{m+j+3}w^\Delta \\ &- \frac{t_1^2}{t_0^2}xz^{m+j+3+\Delta} \pmod{\tilde{I}, s}. \end{aligned}$$

When  $i \leq 3j+1$  we multiply equation (17) with  $z^{3j+1-i}$ . The first term on the right side of this equation is  $xy^2z^{m+2}$  which is  $\equiv 0 \pmod{\tilde{I}, s}$  by  $G_2$ . Note that  $t_1^2 > t_0t_2$ , so the limit is the right term  $xz^{m+4+2j}$  proving a. in this case.

When  $i \geq 3j+2$ , multiplying (16) with  $z^{1+j}$ , the first term on the right is a multiple of  $xy^2z^{m+2}$  which is  $\equiv 0 \pmod{\tilde{I}, s}$ . Hence  $xz^{m+4+2j}$  is in the limit when  $t_2 > t_0$ , and by the results in the first paragraph in this proof, this settles c. When  $t_0 > t_2$  we multiply (16) by  $z^{1+j}$  and get that its left side  $xy^2z^{m+2+2j-i}$  is in the saturated limit. Also the first term on the right of (17) is  $xy^2z^{m+1+i-3j}$  so when  $i \geq 3j+1$  this term is  $\equiv 0 \pmod{\tilde{I}, s}$  and so  $xz^{m+3+i-j}$  is in the saturated limit. This settles b. by Lemma 3.7.  $\square$

**5.4. Monomial orders in general.** In the previous subsections we have assumed monomials containing  $s$  always to be smaller than those without  $s$ . Now we drop this assumption. We do not consider in detail all cases as in the previous subsections, but rather focus on what gives interesting limits. In this subsection we assume  $t_0 > t_1 > t_2$ .

From  $G_{01}$  we obtain when  $i \geq j$  that

$$(18) \quad \begin{aligned} xyz^{m+2+\Delta}w^j &\equiv \frac{t_0t_2}{t_1^2}xyz^{m+2}w^{j+\Delta} - \frac{t_2}{t_1}xz^{m+3+\Delta+j} - \frac{s}{t_1}y^{m+3}z^{1+\Delta+j} \\ &+ \frac{st_0}{t_1^2}y^{m+4}w^{\Delta+j} \pmod{\tilde{I}} \end{aligned}$$

From  $G_{12}$  we obtain

$$(19) \quad xyz^{m+3+j} \equiv \frac{st_1}{t_2^2} y^{m+5} w^j - \frac{s}{t_2} y^{m+4} z^{1+j} \pmod{\tilde{I}}$$

**Proposition 5.7.** *Let  $i \geq j$ . If  $t_2 > s$ , and  $st_1 > t_2^2$ , the limit ideal has  $z$ -transform whose saturation is the Borel ideal with  $\mathbf{a} = (m+2+i-j, m-i)$  and  $\mathbf{b} = (j)$ .*

This proves Part 1. of Theorem 3.3.

*Proof.* Assume first  $t_1^2 > t_0 t_2$ . The following monomials are in the limit.

1.  $xyz^{m+2+\Delta} w^j$  by (18), since  $t_1^2 > t_0 t_2 > t_0 s$ .
2.  $xy^2 z^{m-i}$  by the Basic Limits Lemma.
3.  $y^{m+5} w^j$  by (19).

The  $z$ -transform of the saturated limit then contains

$$xyz^{m+2+i-j}, xy^2 z^{m-i}, y^{m+5} z^j$$

and so by the Saturation Lemma 3.7, we obtain the statement in this case.

When  $t_0 t_2 > t_1^2$  we still have  $t_0 t_2 > t_0 s$ . The only change to the above is that by (18) the limit contains  $xyz^{m+2} w^{\Delta+j}$  instead of  $xyz^{m+2+\Delta} w^j$ , but both have the same  $z$ -transform, giving the result.  $\square$

**Proposition 5.8.** *Let  $j \leq i \leq 2j$ . Assume*

$$t_1 t_2^2 > st_1^2 > t_2^3, \quad t_1^2 > t_0 t_2.$$

*The  $z$ -transform of the limit ideal has as saturation the Borel ideal with  $\mathbf{a} = (m+2+i-2j, m-i)$  and  $\mathbf{b} = (2j)$ .*

This proves Part 2. of Theorem 3.3

*Proof.* Note that  $t_2^2 > st_1$  implies  $t_2 > s$ . The following monomials are in the saturation of the limit.

1.  $xy^2 z^{m-i}$  by the Basic Limits Lemma.
2.  $xyz^{m+2+\Delta} w^j$  by (18).
3.  $xyz^{m+3+j}$  by (19).
4.  $y^{m+5} w^{2j}$  by  $G_{012}^-$  since  $st_1^2$  is greater than  $st_0 t_2$  and  $t_2^3$ .

Parts 2. and 3. above give that  $xyz^{m+2+\Delta}$  is in the saturation. Taking the  $z$ -transform of the ideal, we get the statement.  $\square$

We now consider the case when  $s > t_2$ . This is the only case where we are able to obtain limits with  $p = 1$ .

**Proposition 5.9.** *Assume  $t_0 > t_1 > s > t_2$  and  $st_0 > t_1^2$  and let  $i \geq 2j$ . The limit of the ideal has  $z$ -transform whose saturation is the Borel ideal with  $\mathbf{a} = (m-i+j)$  and  $\mathbf{b} = (i-j, 0)$ .*

*Proof.* The limit contains the following monomials.

1.  $xy^2 z^{m-i} w^i$  by  $G_0$ .

2.  $xy^2z^{m+1-j}w^j$  by  $G_1$ .
3.  $y^{m+5}$  by  $G_2$ .
4.  $y^{m+4}w^{i-j}$  by  $G_{01}$ .

By 1. and 2. the saturation of the limit contains  $xy^2z^{m-i}w^j$ . Taking the  $z$ -transform of the limit, gives the statement.  $\square$

**5.5. The proof of Theorem 3.4.** Having proved Theorems 3.2 and 3.3 as consequences of Propositions 4.1, 5.7 and 5.8, we now prove our third theorem of sufficient conditions.

*Proof of Theorem 3.4.* Part 1. When  $a_1 - a_2 \geq 2$  we may apply Proposition 5.5 b. Let  $j = m + 1 - a_2$ , and  $i + 2 = a_1 - a_2$ . We need to verify the hypothesis  $i \leq j - 1$ . But

$$i = a_1 - a_2 - 2 = a_1 - m - 1 + j - 2 \leq m + 1 - m - 1 + j - 2 = j - 2.$$

When  $a_1 - a_2 = 1$  we may use Proposition 5.6 b. with  $m \geq i = 3j + 2$ . In this case  $\mathbf{a} = (m + 5 + 2j, m + 1 - j, m - j)$ . We may let  $a_1 = m + 1 - j$  which may then be in the range

$$3a_1 = 3m + 3 - 3j = 3m + 5 - (3j + 2) \geq 2m + 5.$$

Part 2. When  $a_1 = m + 2$  we apply Corollary 5.2 b. with  $j = 0$  and  $a_2 = m - i$  with  $i$  in the range  $0 \leq i \leq m$ .

Part 3. Now let  $a_1 = m + 2 + e$  where  $e \geq 1$ . We shall apply Proposition 5.3 b. We may write  $a = pd$  and  $p = 2\rho + 1$ . Let  $j + 1 = \rho d + e$ , and  $i = 2j + d$ . For this to be a valid choice we must show that  $i \leq m$ . But we have

$$\begin{aligned} i = 2j + d &= (2\rho + 1)d + 2e - 2 \\ &= a + 2e - 2 \leq m. \end{aligned}$$

If now  $e < d$  we get  $\Delta = d$  and  $\Delta_2 = e$  and Proposition 5.3 b. applies.

If  $e = d$  we let  $i = a + 2e - 2 \leq m$  and define  $j$  by  $i = 2j + d$ . This is equivalent to letting  $2j = a + e - 2 = (p + 1)d - 2$  which is well defined since the latter is an even number. We then apply Proposition 5.3 a. to get a limit which is the Borel ideal with

$$\mathbf{a} = (m + 2 + e + a, m + 2 + e, m + 2 - 2e - a).$$

$\square$

## 6. A FAMILY OF LIMIT IDEALS WHEN $q = l$

In this section we prove Theorem 3.5. Note that it is a generalization of Corollary 5.2 a. We assume that

$$p_0 \leq p_1 + 1 \leq p_2 + 2 \leq \cdots \leq p_{l-1} + l - 1,$$

and

$$t_0 > t_1 > \cdots > t_{l-1} > s.$$

When

$$\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k (\geq 0)$$

is a partition consisting of  $k$  parts of sizes  $\leq l-1$ , we let  $p_\lambda = \sum p_{\lambda_i}$  and the monomial  $t_\lambda = \prod t_{\lambda_i}$ . We also let  $\bar{\lambda}$  be the complementary partition given by  $\bar{\lambda}_i = l-1-\lambda_{k+1-i}$  for  $i = 1, \dots, k$ .

Let us state the result a bit more elaborately.

**Theorem 6.1.** *Assume for each  $r = 0, \dots, l-1$  that*

1.  $rp_{l-r} \geq p_\lambda$  for all partitions  $\lambda$  of  $r(l-r)$  into  $r$  parts of sizes  $\leq l-1$ .
2.  $t_r^{r+1} > t_\lambda$  for all partitions  $\lambda$  of  $r(r+1)$  into  $r+1$  parts of sizes  $\leq l-1$ .
3. All monomials containing  $s$  are smaller than monomials without  $s$ .

Then the limit ideal has saturation the Borel ideal with  $p = l-1$  and

$$a_i = m + 2(l-1-i) + (l-1-i)p_{i+1} - (l-i)p_i + p_0.$$

*Proof.* Let  $G = [G_0, G_1, \dots, G_{l-1}]^t$  and  $Y = [xy^{l-1}, xy^{l-2}, \dots, x]^t$ . Then  $G = AY + sE$  where  $E = [y^{l+m}, y^{l+m+1}, \dots, y^{2l+m-1}]^t$ , and  $A$  is an  $l \times l$  (symmetric) matrix with rows and columns indexed by  $0, \dots, l-1$ , and the entry in position  $(i, j)$  is

$$t_{i+j} z^{m+i+j-p_{l-1-i-j}} w^{p_{l-1-i-j}}.$$

Let  $A(r)$  for  $1 \leq r \leq l-1$  be the submatrix of  $A$  consisting of the first  $r+1$  rows and  $r$  columns, and let  $M_i$  be the maximal minor of  $A(r)$  obtained by omitting row  $i$ . We may eliminate  $xy^{l-1}, \dots, xy^{l-r}$  from  $G_0, G_1, \dots, G_r$  by forming

$$G_{01..r} = M_0 G_0 - M_1 G_1 + M_2 G_2 + \cdots + (-1)^r M_r G_r.$$

The minor  $M_0$  will be an alternating sum of terms of the form

$$t_{\bar{\lambda}} z^{r(m+r)-p_\lambda} w^{p_\lambda}$$

where the  $\lambda$  are partitions of  $r(l-r-1)$  into  $r$  parts of sizes  $\leq l-1$ . In particular note that the term

$$t_r^r z^{r(m+r)-rp_{l-r-1}} w^{rp_{l-r-1}}$$

occurs as the product of the elements on the anti-diagonals in the submatrix of  $A(r)$  we get by omitting the first row. The terms of  $M_i$  will be a sum of terms

$$t_{\bar{\lambda}} z^{r(m+r)-i-p_\lambda} w^{p_\lambda}$$

where  $\lambda$  is a partition of  $r(l-r-1) + i$  into  $r$  parts of sizes  $\leq l-1$ . Note that if  $i < r$  we can in each term above increase a suitable part of  $\lambda$  to obtain a partition  $\lambda'$  of  $r(l-r-1) + i + 1$ . Then

$$r(m+r) - i - p_\lambda \geq r(m+r) - i - 1 - p_{\lambda'}.$$

So the lowest power of  $z$  will occur in the minor  $M_r$ . By assumption 1. the term with the lowest power in  $z$  is

$$t_{r-1}^r z^{r(m+r-1)-rp_{l-r}} w^{rp_{l-r}}.$$

On the other hand the lowest power of  $w$  occurs in the minor  $M_0$  and this is in the term

$$t_r^r z^{r(m+r)-rp_0} w^{rp_0}.$$

We may then divide each  $M_i$  by the product  $z^{r(m+r-1)-rp_{l-r}} w^{rp_0}$  and let  $M'_i$  be the quotient. Now form

$$G'_{01..r} = M'_0 G_0 - M'_1 G_1 + \dots$$

The question is now what is the limit of  $G'_{01..r}$ . In  $M'_i$  all  $t$ -monomials are of the form  $t_{\bar{\lambda}}$  where  $\bar{\lambda}$  is a partition of  $r(l-r-1) + i$  into  $r$  parts of sizes  $\leq l-1$ . Recall that  $G_i$  consists of terms

$$t_{i+j} xy^{l-j-1} z^{m+i+j-p_{l-1-i-j}} w^{p_{l-1-i-j}}.$$

All the terms containing  $xy^{l-1}, \dots, xy^{l-r}$  are eliminated in  $G'_{01..r}$ . Therefore the largest term in  $M'_i G_i$  which does not get eliminated will occur in

$$(20) \quad M'_i \cdot t_{r+i} xy^{l-r-1} z^{m+r+i-p_{l-1-r-i}} w^{p_{l-1-r-i}}.$$

Now  $\bar{\lambda}$  is a partition of  $r^2 - i$  into  $r$  parts of sizes  $\leq l-1$ . Then  $\bar{\lambda}, r+i$  will be a partition of  $r(r+1)$  into  $r+1$  parts of sizes  $\leq l-1$ . By assumption 2. the largest monomial in the  $t$ 's among the products (20) occurs when  $i = 0$  and

$$\bar{\lambda} : r, r, \dots, r,$$

so

$$\lambda : l-1-r, l-1-r, \dots, l-1-r.$$

Hence the limit term of  $G'_{01..r}$  will be the term occurring in  $M'_0 G_0$ :

$$\begin{aligned} & z^{r+rp_{l-r}-rp_{l-r-1}} w^{rp_{l-r-1}-rp_0} \cdot xy^{l-r-1} z^{m+r-p_{l-r-1}} w^{p_{l-r-1}} \\ &= xy^{l-r-1} z^{m+2r+rp_{l-r}-(r+1)p_{l-r-1}} w^{(r+1)p_{l-r-1}-rp_0}. \end{aligned}$$

When  $r = l-1$  the limit term is

$$xz^{m+2(l-1)+(l-1)p_1-lp_0} w^{p_0}.$$

For each  $r = 1, \dots, l-1$  we multiply this by  $y^{l-r-1}$ . Comparing with (21) we see that the saturation of the limit contains the terms

$$xy^{l-r-1} z^{m+2r+rp_{l-r}-(r+1)p_{l-r-1}} w^{p_0}.$$

This also holds for  $r = 0$  by assigning  $p_l = 0$ , since this will be the limit of  $G_0$ .

Taking the  $z$ -transform it then becomes

$$xy^{l-r-1} z^{m+2r+rp_{l-r}-(r+1)p_{l-r-1}+p_0}.$$

Now the sum of all these powers of  $z$  as  $r = 0, \dots, l-1$  telescopes to  $\sum_{r=0}^{l-1} m + 2r$ , so by the Saturation Lemma 3.7 we obtain the statement of the theorem.  $\square$

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#### REFERENCES

- [1] David Bayer, *The division algorithm and the Hilbert schemes*, Harvard University (1982), Ph.D. Thesis.
- [2] David Bayer and Michael Stillman, *A criterion for detecting  $m$ -regularity*, Invent. Math. **87** (1987), no. 1, 1–11.
- [3] Cristina Bertone, Paolo Lella, and Margherita Roggero, *Borel open covering of Hilbert schemes*, Available at <http://arxiv.org/abs/0909.2184>, 2011, Preprint.
- [4] Francesca Cioffi, Paolo Lella, Maria Grazia Marinari, and Margherita Roggero, *Segments and Hilbert schemes of points*, Discrete Mathematics **311** (2011), no. 20, 2238 – 2252.
- [5] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.
- [6] Geir Ellingsrud, *Sur le schéma de Hilbert des variétés de codimension 2 dans  $\mathbf{P}^e$  à cône de Cohen-Macaulay*, Ann. Sci. École Norm. Sup. (4) **8** (1975), no. 4, 423–431.
- [7] R. Hartshorne, *Algebraic geometry*, vol. 52, springer Verlag, 1977.
- [8] M. Kreuzer and L. Robbiano, *Computational commutative algebra*, vol. 2, Springer, 2005.
- [9] Paolo Lella, *BorelGenerator*, Available at <http://www.dm.unito.it/dottorato/dottorandi/lella/HSC/borelGenerator.html>.
- [10] ———, *A network of rational curves on the Hilbert scheme*, Available at <http://arxiv.org/abs/1006.5020v2>, 2010, Preprint.
- [11] Alyson Reeves and Mike Stillman, *Smoothness of the lexicographic point*, J. Algebraic Geom. **6** (1997), no. 2, 235–246.
- [12] Alyson A. Reeves, *The radius of the Hilbert scheme*, J. Algebraic Geom. **4** (1995), no. 4, 639–657.

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